The Inefficiency of Contractually-Based Liability with Rational Consumers

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Abstract

The prevailing view in the law and economics literature has long been that preventing firms and consumers from contracting out of mandatory product liability rules can only be justified if consumers are, in some way, irrational or misperceive the risks of the products they buy. In this paper, I take the contrary view that even if consumers do correctly judge the risks of the products they buy, if they cannot directly observe the safety characteristics of the products they buy, then allowing firms and consumers to write their own liability rules cannot lead firms to make efficient investments in product safety (unless the efficient level of investment is zero). Because the legal system is costly, consumers always have an incentive to waive liability in exchange for a lower price after safety investments are sunk. If they do so, however, firms will anticipate this, thereby undermining there incentive to invest in safety. Only mandatory product liability provides a mechanism for consumers and firms to commit not waive liability.
1 Introduction

The prevailing view in the law and economics literature has long been that preventing firms and consumers from contracting out of mandatory product liability rules can only be justified if consumers are, in some way, irrational or misperceive the risks of the products they buy. As Spence (1977) put it, "The mere fact that products may fail or cause accidents is not by itself an argument for intervention in the market. But, because of the random character of the outcome, there is a suspicion that consumers are not accurately informed about the distribution of possible outcomes prior to purchase." This view has been echoed in many subsequent economic analyses of product liability (see, for example, Shavell (2003) and Geistfeld (1995)). In fact, this view has led many commentators to argue that mandatory product liability should be eliminated or curtailed in favor of contractually-based liability (Rubin 1999; Priest 1992; Huber 1988). Furthermore, many states have enacted statutes that limited consumers ability to recover for product-caused injuries, forcing them to rely more on contractual remedies for such losses (Geistfeld 1994).

In this paper, I take the contrary view that even if consumers do correctly judge the risks of the products they buy, if they cannot directly observe the safety characteristics of the products they buy, then allowing firms and consumers to write their own liability rules cannot lead firms to make efficient investments in product safety (unless the efficient level of investment is zero). What drives this result is, paradoxically, the inefficiency of the legal system. Imagine that there are two ways to make a water heater, the unsafe way and the safer way. Naturally, the safer way costs more than the unsafe way. Furthermore, assume that a consumer purchasing the water heater cannot observe whether the firm used the unsafe or the safer way. Thus, if the firm is not liable for water heater accidents, it has no incentive to use the more costly, but safer, manufacturing process. One might think that if the cost making the water heater safer is less than the benefit from the increased safety (a lower accident probability), that firms and consumers would write a contract that holds the firm liable for accidents. If using the tort system to provide consumers with insurance against injuries is more costly than purchasing first party insurance directly, however, this cannot always be the case. To see this, imagine that when firms are choosing their manufacturing process, they assume their contract with consumers will hold them liable for injuries. As a result, they use the safer, but more costly method. At the time a consumer is buying a water heater, however, the manufacturing process has already been chosen. If, in equilibrium, firms always use the safer method, the consumer can offer to waive liability and purchase the water heater at a lower price.
and use the savings to purchase first party insurance and (because the legal system is inefficient) have money left over. But, since this is the optimal strategy for the consumer, the firm would anticipate that it would not be held liable. Thus, it would have no incentive to use the safer method.

This result is a variation of Fudenberg and Tirole’s (1990) result that renegotiation undermines second best agency contracts. In that model, the agent’s wage must vary with output to induce the agent to work hard. The cost of such a contract is that it exposes the risk averse agent to risk. But, if the principal and the agent can renegotiate after the agent has chosen her effort level, then both sides can be made better off if the principal (who is risk neutral) replaces the agent’s incentive contract with a flat wage. Since the agent’s effort level is now fixed, there are no incentive costs, only risk-sharing benefits. But, if the agent anticipates this renegotiation will take place, she has no incentive to work hard in the first place.\(^1\)

In my model, risk is not an issue since the consumer can purchase first party insurance, so, I assume for simplicity, that the consumer and firm are risk neutral. Instead, the cost of the legal system plays the role that risk aversion plays in the standard agency model; it is what makes the optimal incentive contract ex post inefficient. Thus, as in Fudenberg and Tirole’s model, the only way to sustain an ex post inefficient incentive contract is if there is also some ex post asymmetric information. That is, the firm can not make product safer with probability one. Instead, even if the benefits of safety exceed the cost, there is a maximum probability that the firm can make the product safer. The reason is that if the likelihood that the product is safer is too high, the consumer prefers to offer a contract that waives liability in exchange for a lower price that both a firm that makes a safer and an unsafe product prefer to the full liability contract. But, again, if the consumer does so, then the firm has no incentive to make the product safer. As in agency model, the problem is lack of commitment. Both parties would be better off if they could commit ex ante not to waive liability, but under contractually-based liability, such a commitment is not possible. Mandatory liability provides such a commitment.

Of course, this only establishes that contractually-based liability has an important disadvantage relative to mandatory liability—it cannot induce the firm to make costly and unobservable safety

\(^1\)It should be noted that Fudenberg and Tirole’s result depends on the principal making the renegotiation offer. Ma (1994) did show that if the agent, rather than the principal, makes the renegotiation offer, then it is possible to sustain the second best outcome with renegotiation. Wickelgren (2003), however, shows that if both parties can make renegotiation offers, then the Fudenberg and Tirole (1990) result is restored. A similar caveat applies to the results here; they are valid so long as the consumer has the opportunity to propose waiving liability, even if the firm can also propose a liability contract.
investments with probability one. Mandatory liability, because it provides a commitment device that prevents the harmful effects of renegotiation, of course, can induce the firm to make such investments. This means that mandatory product liability is strictly superior to contractual-based liability if making the product safe is necessarily efficient. If, however, the cost of making the product safe is too high, or the legal system is too costly, then welfare could be greater without these safety investments but without the costs of legally enforced liability as well. The advantage of contractually-based liability is that it allows the firm to use its information about the cost of making the product safer to determine whether such safety investments are efficient. Contractually-based liability avoids the costs of liability when these investments are inefficient, while still making these investments with some probability (though, not with probability one) when these investments are efficient.

So, whether mandatory or contractually-based liability is optimal will depend on the probability that making the product safer is efficient and on the cost of the legal system. Not surprisingly, the greater the likelihood that the cost of making the product safer is low, the greater the advantage of mandatory liability. There are, however, two somewhat surprising comparative static results. These are that both the cost of the legal system and the probability of a safer product causing an accident actually have a non-monotonic effect on the relative desirability of mandatory liability. When both of these are very low, increasing each of them actually increases the desirability of mandatory liability. The reason is that the product of the cost of the legal system and the probability of a safer product causing an accident gives the expected legal costs when the product is safer. If these expected legal costs are very low, there is little ex post inefficiency associated with holding the firm liable. As a result, the maximum probability that the firm will invest in safety under contractually-based liability is greater, reducing the relative incentive advantage of mandatory liability over contractually-based liability. As expected legal costs rise, contractually-based liability becomes less effective at inducing the firm to invest in safety (the maximum probability declines), making mandatory liability relative more advantageous. But, if expected legal costs get too large, then the transactions costs of mandatory liability start to outweigh the incentive benefits, making mandatory liability less advantageous.

The next section describes the model, first for a constant cost of investing in safety, so that investing in safety is necessarily efficient, then when there is some probability that investing in safety is inefficient. The last section concludes. Proofs not contained in the text are in the Appendix.
2 Model

2.1 Constant c

A firm offers a product to a risk neutral consumer, who values the product a $v$. The firm can invest in making the product safer at a cost of $c$ or not. All additional production costs are normalized to zero. If the firm invests in safety, the probability of an accident is $q_l$, if it does not the probability is $q_h$; $q_l < q_h$. Neither the consumer nor the courts can observe whether the firm invested in safety. The harm suffered by the consumer if there is an accident is $h$. If the firm is liable, then the consumer and the firm each spend $k_c$ and $k_f$ respectively on legal costs if an accident occurs. Let $k = k_c + k_f$ be the total legal costs from going to court.

I assume that $v > q_h(h + k)$, so the consumer always values the product more than the expected safety and litigation costs, even if the firm does not invest in safety. Furthermore, assume that $c < (q_h - q_l)h$; investing in safety is always efficient (I relax this assumption in the next subsection). Clearly, the first best solution is for the firm to invest in safety and for it not to be liable for damages in the event of an accident. Since investing in safety is unobservable, however, the first best is not achievable. I now consider what happens if the consumer and the firm are free to write their own contract. I assume that they contract after the firm has made the decision of whether or not to invest in safety. The results would be similar if they could contract before the firm made its investment decision as long as they had the opportunity to renegotiate this contract after the safety decision has been made.

If the parties are allowed to contract out of strict product liability, then the optimal contract will consist of the following menu. Each contract will specify whether or not the firm is liable in the event of an accident and two prices, one for each liability regime. Thus, the consumer will offer the firm the following menu of contracts is $\{(p^N, N), (p^L, L)\}$. $p^j$ is the price for the good in liability regime $j$ ($N$ for not liable, $L$ for liable). The second element of each pair ($N$ or $L$) specifies the liability regime.

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2I assume that the consumer cannot affect the probability of an accident. To the extent that consumer efforts are an important determinant of accident probability, this provides an argument against full producer mandatory liability, but does not undermine the results in this paper about the limitations of contractually-based liability. See Priest (1981) for a discussion of the importance of consumer precautions and Cooper and Ross (1985) for a formal model of warranties with double moral hazard.

3In principle, one could allow for partial liability. But, since I have assumed fixed costs of going to court, partial liability will never be optimal. Similarly, given that consumers and firms are risk neutral and consumers cannot affect the probability of an accident, contracts that specify less than full compensation in the event of an accident are also suboptimal.

4As discussed in the introduction, the assumption that the consumer make make a contract offer to the firm is
Assume that there is free entry into this market, so the firm must earn zero expected profits. Since the safety investment is sunk before the consumer and firm contract, this implies that a firm that invests in safety must have some ex post market power, otherwise it would not be able to recoup the fixed cost of its investment. A firm that invests in safety will make zero profits with either a contract of \((c, N)\) (the price just covers the cost of its investment in safety) or a contract of \((c + q_l(h + k_f), L)\) (the price covers the cost of the safety investment and its expected legal costs).\(^5\) A firm that does not invest in safety will make zero profits with either a contract of \((0, N)\) or a contract of \((q_h(h + k_f), L)\). Now consider the legal regime where the default rule is strict liability but the customer is allowed to offer a no liability contract. Thus, under the default rule of firm liability, I assume the firm has enough market power to sell the good to the consumer for a price of \(c + q_l(h + k_f)\) as long as the consumer believes the firm invested in safety.\(^6\) Without this assumption, there is no way to get the firm to ever invest in safety.

For any given type of firm, the customer is always better off with the no liability contract. If the firm invests in safety and the contract gives it zero profits, the customer’s expected utility is \(v - c - q_l h\) with the no liability contract and is \(v - c - q_l(h + k)\) with the liability contract. Similarly, if the firm does not invest in safety and the contract gives it zero profits, the customer’s expected utility is \(v - q_h h\) with the no liability contract and is \(v - q_h(h + k)\) with the liability contract. Thus, if the customer can correctly infer the type of firm, she will always choose to offer the firm a no liability contract that gives the firm the same expected profits it would receive with the default liability contract. Anticipating this, however, the firm would never invest in safety.

That is, the only pure strategy equilibrium that exists is where the firm does not invest in safety and the consumer offers to waive liability and pay zero rather than \(q_h(h + k_f)\) for the good. Since investing in safety is efficient, however, this equilibrium is inferior to any mixed strategy equilibrium that may exist. I now check to see that such an equilibrium exists where the firm invests in safety with probability \(\pi\). In this equilibrium, the consumer will offer the firm a choice from the following menu of contracts \{\((0, N), (c + q_l(h + k_f), L)\}\}. With this menu, if the firm

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\(^5\)I assume liability equals harm, \(h\). Given the binary nature of safety investment, and that investing in safety is always efficient, this is one of many optimal liability rules.

\(^6\)One way to think about this is to say that there are many firms who enter without investing in safety, but only a small number who enter and invest in safety. Thus, the safe firms, if the consumer can indentify them, have some (ex post) market power. The reason that only a small number of firms enter and invest in safety is because doing so is costly. So an individual firm can only recoup its investment if only a few other firms enter.
invests in safety it strictly prefers the liability contract and if it did not invest in safety, it strictly
prefers the no liability contract. The consumer’s utility of offering this menu is:

\[ v - \pi(c + ql(h + k)) - (1 - \pi)qh \]  

(1)

The firm is clearly indifferent between investing in safety or not ex ante since it earns zero ex ante
profits either way. For this to be an equilibrium, the consumer must not want to deviate by
increasing \( p^N \) to get the firm that invests in safety to choose the no liability contract. For a safe
firm to choose the no liability contract, the consumer must offer \( p^N = c \). If it does so, however,
then it must pay \( c \) to both the safe and the unsafe firm. If the probability that the firm is safe is
\( \pi \), this gives the consumer an expected utility of:

\[ v - c - \pi qlh - (1 - \pi)qh \]  

(2)

Deviating then lowers the consumer’s utility if and only if (2) \( \leq \) (1), that is if and only if:

\[ \pi \leq \pi^* = \frac{c}{qh} \]  

(3)

This \( \pi^* \) represents the maximum equilibrium probability of the firm investing in safety when the
law allows the consumer and firm to contract out of firm liability. Thus, I have proved the following
proposition.

**Proposition 1** If the firm and the consumer can contract out of product liability, then the proba-
bility that the firm invests in safety is no greater than \( \frac{c}{qh} < 1 \). That is, even if investing
in safety is efficient, the firm will not invest in safety with probability one if the consumer
and firm are free to write their own contract.

When safety investment is unobservable, and firms and consumers have the opportunity to
contract after the product has been made, so that any safety investments have already been made,
there is no way to induce the firm to always make a safe product. Notice that this result is very
similar to Fudenberg and Tirole’s (1990) result an agent cannot be induced to work hard with
probability one if the principal and the agent can renegotiate their contract after the agent has
chosen her effort level. In this model, the consumer is the principal and the firm is the agent.
Litigation costs play the same role that risk aversion plays in Fudenberg and Tirole’s model: the
ex post efficient contract eliminates litigation costs (rather than the agent’s risk), but doing so eliminates ex ante incentives. The only way to maintain the firm’s incentives so that it always invests in safety is for the firm and the consumer to be able to commit not to contract out of the liability rule after the product has been made. Mandatory strict liability provides this commitment device.

I now evaluate the consumer’s utility when she can contract out of mandatory strict liability. I assume that the firm invests in safety with the largest possible probability, so I evaluate (1) at \( \pi = \pi^* \). Doing so gives the following:

\[
v - c - q_h h + \frac{(q_h - q_l)hc}{q_lk + c}
\]

(4)

I now compare this to utility where the firm and the consumer are not allowed to contract out of product liability. (Since firm profits are zero in either case, this is also a total welfare comparison.) In this case, the firm invests in safety and charges the consumer \( c + q_l(h + k_f) \) for the good. If there is an accident, the consumer has to spend \( k_c \) to go to court to get the firm to pay for the consumer’s harm. Thus, the consumer’s utility is:

\[
v - (c + q_l(h + k))
\]

(5)

By subtracting (4) from (5), one can determine the consumer’s gain from mandatory product liability:

\[
q_lk \frac{(q_h - q_l)h - (q_lk + c)}{q_lk + c}
\]

(6)

Thus, I have proved the following proposition.

**Proposition 2** Mandatory product liability strictly increases consumer utility and total welfare if and only if \( (q_h - q_l)h > (q_lk + c) \), that is, if and only if the benefits of investing in safety exceed its costs plus legal costs.

Given that the free entry ensures that the firm earns zero profits in equilibrium, the more efficient rule will also benefit the consumer the most. Thus, it is not surprising that the consumer benefits from mandatory product liability if and only if the incentive benefits of the rule outweigh
the costs. The next proposition, however, does describe some surprising results about how the total welfare gain from mandatory product liability varies with the different parameters of the model.

**Proposition 3** The gain in total welfare from mandatory product liability is (i) decreasing in \(c\) (ii) increasing in \(h\) (iii) increasing in \(q_h\) (iv) increasing (decreasing) in \(q_l\) if \(q_l < (>) \sqrt{ch(q_h + kq_l)}\) (v) increasing (decreasing) in \(k\) if \(k < (>) \sqrt{ch(q_h - q_l) - c}/q_l\).

Proof. See Appendix.

The first three results are exactly what one would expect. The less costly investing in safety is, the greater the gain from mandatory product liability that induces more investment in safety. Similarly, the greater the harm from an accident or in the chance of the unsafe product causing an accident, the more valuable it is to induce the firm to invest in safety. One might also think that the smaller the probability of a safe product causing an accident, the more the consumer would benefit from mandatory product liability. The reason this is not always true is that \(\pi^*\) (the maximum probability that the firm will invest in safety without mandatory liability) is decreasing in \(q_l\). When \(q_l\) is very small, the ex post inefficiency from liability is quite small since the consumer is very unlikely to have to go to court. This makes deviation to a no liability contract less profitable for any given \(\pi\), increasing the maximum feasible \(\pi\). So, if \(q_l\) is very small, the difference in the probability that the firm invests in safety between mandatory liability and contractual based liability is quite small, making the advantage of mandatory product liability quite small. Once \(q_l\) is large, however, the difference in the probability that the firm invests in safety becomes substantial. At this point, since larger \(q_l\) makes investing in safety less valuable, it decreases the benefit of mandatory product liability. The same reasoning explains the non-monotonic effect of \(k\) as well. When \(k\) is small, liability is not very inefficient, so \(\pi^*\) can be large. But if \(k\) gets too large, then the ex post inefficiency of liability starts to significantly counteract the incentive benefits of mandatory liability. Thus, the argument in favor of mandatory liability is strongest for intermediate values of legal costs.

### 2.2 Variable \(c\)

In the last subsection, the cost of investing in safety was fixed so that investing in safety was always efficient. While this model is sufficient to demonstrate that rational parties cannot necessarily achieve the efficient outcome when they have freedom to contract, it also assumes away any benefit from giving parties the flexibility to choose their own liability rule. In this subsection, I assume
that $c$ can take one of three values, $c_L$, $c_M$, or $c_H$, and that the probability of each is $r_L$, $r_M$, and $r_H = 1 - r_L - r_M$ respectively.\(^8\) I make the following assumption about the three values of $c$:

$$c_L < (q_h - q_l)h - q_lk < c_M < (q_h - q_l)(h + k) < c_H \tag{7}$$

Thus, if $c \in \{c_M, c_H\}$, then holding the firm liable is inefficient because the cost of investing in safety plus the expected legal costs of liability exceed the reduction in the expected harm from using the product. If $c = c_M$, however, the firm will still invest in safety under mandatory liability since the cost of investing safety is less than difference in expected liability plus the difference in expected legal costs. For $c = c_H$ the firm will not invest in safety under either regime. For either of these values of $c$, allowing the parties to contract out of firm liability will be superior to mandatory product liability.

Because the firm earns zero profit (due to free entry), when $c \in \{c_M, c_H\}$ the consumer’s utility is greater when the firm does not invest in safety. So, in the best equilibrium for the consumer (and for total welfare), $\pi = \pi^*$ if $c = c_L$ and $\pi = 0$ if $c \in \{c_M, c_H\}$. The consumer’s expected utility from being able to contract out of product liability is:

$$r_L(v - c_L - q_hh + \frac{(q_h - q_l)hc_L}{qh k + c_L}) + (1 - r_L)(v - q_hh) \tag{8}$$

Simplifying, this becomes:

$$v - q_hh + r_Lc_L\left(\frac{(q_h - q_l)h}{qh k + c_L} - 1\right) \tag{9}$$

With mandatory product liability, the firm will invest in safety if $c \in \{c_L, c_M\}$, the cost of investing in safety is less than the expected benefit of reducing the expected harm and the expected legal costs. In this regime, the consumer’s expected utility is:

$$r_L[v - (c_L + q_l(h + k))] + r_M[v - (c_M + q_l(h + k))] + (1 - r_L - r_M)(v - q_h(h + k)) \tag{10}$$

Simplifying, this becomes:

$$v - q_h(h + k) + (r_L + r_M)(q_h - q_l)(h + k) - r_Lc_L - r_Mc_M \tag{11}$$

\(^8\)The case where $c$ is a random variable that can take any value between $c$ and $\pi$ is quite similar and does not yield any significant further insights.
Subtracting (9) from (11) gives the expected consumer (and welfare) gain from mandatory product liability. This difference is:

\[
\frac{r_L q_l k}{q_l k + c_L}((q_h - q_l)h - q_l k - c_L) - r_M(c_M - (q_h - q_l)h + q_l k) - (1 - r_L - r_M)q_h k
\]

(12)

By (7), the first term is positive and the next two terms are negative. (The first term is exactly \(r_L \ast (6)\), the social welfare gain when \(c\) is low with probability one.) So, not surprisingly, mandatory product liability is benefits consumers when the cost of investing in safety is very likely to be low and hurts consumers when there is a large probability that the cost of investing of safety is medium or large.

The comparative statics with uncertain costs of investing in safety are also very similar to what they were in the constant \(c\) case, as the next proposition indicates.

Proposition 4 The gain in total welfare from mandatory product liability is (i) increasing in \(r_L\) and decreasing in \(r_M\) and \(r_H = 1 - r_L - r_M\) (ii) increasing in \(h\) (iii) increasing in \(q_h\) if and only if \(r_H < \frac{r_M}{r_L}\) (iv) increasing in \(q_l\) if and only if \(q_l < \frac{q_H}{q_L}\) for some \(q_H\) (v) increasing in \(k\) if \(k < \frac{q_H}{q_L}\) for some \(k\).

Proof. See Appendix.

Not surprisingly, the greater the harm from the accident, the more important it is to induce investment in safety, which mandatory product liability is more effective at doing. A similar argument explains why larger \(q_h\) makes the gain from mandatory product liability greater. The reason this result does not hold for large \(r_H\) is that when the cost of investing in safety is high, there is no investment in safety with either mandatory liability or voluntary contracting. The only difference is that the customer must pay legal costs with mandatory product liability, whose expected level are greater the greater the probability of an accident. The reason that the welfare gain from mandatory product liability is increasing in \(q_l\) and \(k\) when these values are small is that, as we saw in the prior subsection, these parameters determine the maximum probability of investing in safety with voluntary contracting. When there is very little probability of an accident when the firm invests in safety or if legal costs are very small, expected legal costs from liability are small. This means that the maximum probability that the firm will invest in safety under voluntary liability can be quite large, mitigating the advantage of mandatory liability. As these parameters get bigger, the maximum probability the firm will invest in safety with voluntary contracting becomes quite small, making the difference in the incentive effects of the two regimes
quite large. When expected legal costs get too large, however, then they begin to overwhelm the incentive advantages of mandatory liability. Once again, the argument for mandatory liability is greatest with intermediate values of expected legal costs.

3 Conclusion

This paper demonstrates that the widely held belief that consumers must systematically misperceive the risks associated with a product for mandatory product liability to be optimal is incorrect if consumers cannot observe investments in safety. In this model, consumers do rationally infer the expected accident probability of a product, and, even offer screening contracts to be able to perfectly distinguish safer products from unsafe products. Nonetheless, under voluntary contracting, there is no equilibrium where firms invest in safety with probability one even when doing so is efficient. Because using the legal system is costly, consumers have an incentive, once safety investments are sunk, to waive liability and obtain lower prices. To prevent them from doing so, and thereby completely undermining a firm’s incentive to invest in safety, the consumer must believe that there is some minimum probability that the firm did not invest in safety. The only way to induce the firm to invest in safety with probability one is to make liability mandatory. That is, the inefficiency of the legal system creates an argument for making using the legal system mandatory.

While this paper does not show that mandatory product liability is necessarily more efficient than contractually based liability, it does demonstrate that the simple argument that rational consumers and firms can achieve through contract the same efficient outcomes that mandatory liability can is mistaken. Instead, call for restricting mandatory liability must acknowledge that there is a real incentive cost to doing so, even if consumers are completely rational. There maybe situations where the benefits of restricting liability in terms of reduced legal costs outweigh the loss in incentives for investments in safety, but such a comparison of costs and benefits cannot be avoided. Contractually-based liability cannot achieve the first, or even second, best.
APPENDIX

Proof of Proposition 3. (i) Differentiating \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \) with respect to \( c \) gives \( -q_k \frac{(q_n - q_l)}{(q_k + c)^2} \) < 0. (ii) Differentiating \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \) with respect to \( h \) gives \( q_k \frac{(q_n - q_l)}{(q_k + c)^2} > 0 \). (iii) Differentiating \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \) with respect to \( q_h \) gives \( \frac{q_k h}{q_k + c} > 0 \). (iv) Differentiating \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \) with respect to \( q_l \) gives \( k \left( -\frac{c}{k + q_n} \right) \). This has the sign of \( c(h(q_n - q_l) - (q_k + c)(c + (h + k)q_l) \). Taking the derivative of this with respect to \( q_l \) gives \( -2(h + k)(q_k + c) < 0 \), so \( c(h(q_n - q_l) - (q_k + c)(c + (h + k)q_l) \) is strictly decreasing in \( q_l \). Solving \( c(h(q_n - q_l) - (q_k + c)(c + (h + k)q_l) = 0 \) gives \( q_l = \frac{\sqrt{c(h(q_n - q_l) - c)}}{k} \). (v) Differentiating \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \) with respect to \( k \) gives \( q_k \frac{(q_n - q_l)(q_n - (q_k + c))}{q_k + c} \). This has the sign of \( c(h(q_n - q_l) - (q_k + c)^2) \). Taking the derivative of this with respect to \( k \) gives \( -2q_k^2(q_k + c) < 0 \), so \( c(h(q_n - q_l) - (q_k + c)^2) \) is strictly decreasing in \( k \). Solving \( c(h(q_n - q_l) - (q_k + c)^2) = 0 \) gives \( k = \frac{\sqrt{c(h(q_n - q_l) - c)}}{q_k} \). Q.E.D.

Proof of Proposition 4. (i) Write (12) as \( r_L q_k \frac{(q_n - q_l)(q_n - q_k - c_L)}{q_k + c_L} - r_M(c_M - (q_n - q_l)h + q_k) - r_Hq_kh_k \). The first term is \( r_L \neq (6) \) and we know that \( (6) > 0 \). \( r_M \) only appears in the second term, which is negative by the definition of \( c_M \). The third term is where \( r_H = 1 - r_L - r_M \) appears and that is clearly decreasing in \( r_H \). (ii) Differentiating (12) with respect to \( h \) gives \( \frac{(r_L + r_M)c_L}{q_k + c_L} q_k \frac{(q_n - q_l)h - q_k - c_L} {q_k + c_L} > 0 \). (iii) Differentiating (12) with respect to \( q_h \) gives \( r_M h - k(1 - r_L - r_M - \frac{r_L q_k h}{q_k + c_L}) = r_M h - k(r_H - \frac{r_L q_k h}{q_k + c_L}) \) which is positive if and only if \( r_H < h(\frac{r_L q_k h}{q_k + c_L} + \frac{r_L}{k}) \). (iv) Differentiating (12) with respect to \( q_l \) gives \( k(\frac{c_L^2 - (h + k)q^2 + c_L(h(q_n - q_l) - 2q_l) - 2kq_l + r_L - (h + k)(q_k + c_L)^2 r_M}{(q_k + c_L)^2}) \). This has the sign of the numerator. Differentiating the numerator with respect to \( q_l \) gives \( -2k(h + k)(q_k + c_L)(r_L + r_M) < 0 \). So, the numerator is positive only for small \( q_l \). Evaluating it at \( q_l = 0 \) gives \( c_L(k(q_n - q_l) + c_L)r_L - c_L(h + k)r_M \). This is positive if \( r_L \) is large relative to \( r_M \). (v) Differentiating (12) with respect to \( k \) gives \( \frac{r_L c_L h(q_n - q_l)q_k}{(q_k + c_L)^2} - q_h(1 - r_L - r_M) - q_l(r_L + r_M) \). This is clearly decreasing in \( k \), so if it will ever be positive it will be positive for small \( k \). At \( k = 0 \), it is \( \frac{r_L h(q_n - q_l)q_k}{c_L} - q_h(1 - r_L - r_M) - q_l(r_L + r_M) \). This is positive if \( r_L \) is large enough; at \( r_L = 1 \) it is \( \frac{r_L q_k}{c_L}(h(q_n - q_l) - c_L) > 0 \). Q.E.D.
References


